

Input-to-state stability of infinite-dimensional control systems

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Abstract We develop tools for investigation of input-to-state stability (ISS) of infinite-dimensional control systems. We show that for certain classes of admissible inputs the existence of an ISS-Lyapunov function implies the input-to-state stability of a system. Then for the case of the systems described by abstract equations in Banach spaces we develop two methods of construction of local and global ISS-Lyapunov functions. We prove a linearization principle that allows a construction of a local ISS-Lyapunov function for a system which linear approximation is ISS. In order to study interconnections of nonlinear infinite-dimensional systems, we generalize the small-gain theorem to the case of infinite-dimensional systems and provide the way to construct an ISS-Lyapunov function for an entire interconnection, if a ISS-Lyapunov functions for subsystems are known and the small-gain condition is satisfied. We illustrate theory on examples of linear and semilinear reaction-diffusion equations.

Keywords nonlinear control systems · infinite-dimensional systems · input-to-state stability · Lyapunov methods · linearization

1 Introduction

The concept of input-to-state stability (ISS) introduced in [30] is widely used to study stability properties of control systems with respect to external inputs.

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Within last two decades different methods for verification of the input-to-state stability of finite-dimensional systems were developed. For a survey of recent results in the ISS theory see [32] and [5], the connection of ISS and circle criterion can be found in [18]. In particular it is known that the method of Lyapunov functions together with small-gain theorems (see [19], [8], [10], [20]) provides us with rich tools to investigate input-to-state stability of control systems.

Apart from systems based on ordinary differential equations the ISS concepts were applied to hybrid, switched and impulsive systems: [16], [36], [2]. But for input-to-state stability of infinite-dimensional systems, with an important exception of time-delay systems (see, e.g. [27], [20]), less attention was devoted. In [6] some basic results for certain classes of reaction-diffusion systems were presented. In the paper [23] the ISS of semilinear parabolic equations has been studied with the help of strict Lyapunov functions.

We study both local and global ISS of infinite-dimensional control systems.

Our first main result is that for abstract control systems under certain assumptions on the class of input functions from the existence of a (local or global) ISS-Lyapunov function it follows (local or global) ISS of the system. We show that our definition of the local ISS-Lyapunov function is consistent with the standard definition of local ISS-Lyapunov function for finite-dimensional systems.

In the next part of the paper we exploit semigroup theory methods and consider infinite-dimensional systems generated by differential equations in abstract spaces. For such systems we develop two methods for construction of ISS-Lyapunov functions for the control systems.

To study interconnections of n ISS subsystems, we generalize small-gain theorem for finite-dimensional systems [7], [10] to the infinite-dimensional case. This theorem allows a construction of a Lyapunov function for the whole interconnection if the Lyapunov functions for subsystems are known and the small-gain condition is satisfied. The ISS of the interconnection follows then from the existence of the Lyapunov function for it.

The local ISS of nonlinear control systems can be investigated in an analogous way (see, e.g., [9]), but also another type of results is possible, namely linearization technique, well-known for infinite-dimensional dynamical systems (without inputs) [15]. We prove a linearization theorem for nonlinear control systems over Hilbert spaces, which provides a form of a LISS-Lyapunov function for a nonlinear systems, if the linear approximation of the system is ISS.

Throughout the paper we use either classical solutions of partial differential equations, or the solutions in the Sobolev spaces. Another function spaces can be also exploited, see e.g. [4].

The outline of the work is as follows: in Section 2 we introduce notation and basic notions. In Section 3 we discuss ISS for linear systems. Afterwards the method of ISS-Lyapunov functions is extended to the abstract control systems and the results are applied to the certain nonlinear reaction-diffusion equation. In Section 5 we prove the linearization principle. Next, in Section

6 we prove a small-gain theorem and apply it to certain linear and nonlinear systems. In Section 7 we conclude the results of the paper.

2 Preliminaries

Throughout the paper let $(X, \|\cdot\|_X)$ and $(U, \|\cdot\|_U)$ be a state space and space of input values, endowed with norms $\|\cdot\|_X$ and $\|\cdot\|_U$ respectively. For definiteness let them be Banach spaces.

For Banach spaces X, Y let $L(X, Y)$ be the spaces of bounded linear operators from X to Y and $L(X) := L(X, X)$. A norm in these spaces we denote by $\|\cdot\|$.

By $C(X, Y)$ we denote the space of continuous functions from X to Y , $C(X) := C(X, X)$ and by $PC(X, Y)$ the space of piecewise right-continuous functions from X to Y . Both are equipped with the standard sup-norm.

Let $\mathbb{R}_+ := [0, \infty)$. We will use throughout the paper the following function spaces:

- $C_0^k(0, d)$ is a space of k times continuously differentiable functions $f : (0, d) \rightarrow \mathbb{R}$ with compact in $(0, d)$ support.
- $L_p(0, d)$ is a space of p -th power integrable functions $f : (0, d) \rightarrow \mathbb{R}$ with the norm $\|f\|_{L_p(0, d)} = \left(\int_0^d |f(x)|^p dx \right)^{\frac{1}{p}}$.
- $W^{p, k}(0, d)$ is a Sobolev space of functions $f \in L_p(0, d)$, which have weak derivatives of order $\leq k$, all of which belong to $L_p(0, d)$.
- $W_0^{p, k}(0, d)$ is a closure of $C_0^k(0, d)$ in the norm of $W^{p, k}(0, d)$.
- $H^k(0, d) = W^{2, k}(0, d)$, $H_0^k(0, d) = W_0^{2, k}(0, d)$.

We use the following axiomatic definition of the control system

Definition 1 *The triple $\Sigma = (X, U_c, \phi)$, consisting of state space X , linear normed space of admissible input functions $U_c \subset \{f : \mathbb{R}_+ \rightarrow U\}$ (with the norm $\|\cdot\|_{U_c}$) and of a transition map $\phi : A_\phi \rightarrow X$, $A_\phi \subset \mathbb{R}_+ \times \mathbb{R}_+ \times X \times U_c$ is called a control system, if the following properties hold:*

- *Existence:* For every $(t_0, \phi_0, u) \in \mathbb{R}_+ \times X \times U_c$ there exists $t > t_0$: $[t_0, t] \times \{(t_0, \phi_0, u)\} \subset A_\phi$.
- *Identity property:* For every $(t_0, \phi_0) \in \mathbb{R}_+ \times X$ it holds $\phi(t_0, t_0, \phi_0, \cdot) = \phi_0$.
- *Causality:* For every $(t, t_0, \phi_0, u) \in A_\phi$, for every $\tilde{u} \in U_c$, such that $u(s) = \tilde{u}(s)$, $s \in [t_0, t]$ it holds $(t, t_0, \phi_0, \tilde{u}) \in A_\phi$ and $\phi(t, t_0, \phi_0, u) \equiv \phi(t, t_0, \phi_0, \tilde{u})$.
- *Continuity:* for each $(t_0, \phi_0, u) \in \mathbb{R}_+ \times X \times U_c$ the map $t \rightarrow \phi(t, t_0, \phi_0, u)$ is continuous.
- *Semigroup property:* for all $t, s \geq 0$, for all $\phi_0 \in X$, $u \in U_c$ so that $(t, s, \phi_0, u) \in A_\phi$, it follows
 - $(r, s, \phi_0, u) \in A_\phi$, $r \in [s, t]$,
 - for all $r \in [s, t]$ it holds $\phi(t, r, \phi(r, s, x, u), u) = \phi(t, s, x, u)$.

Here $\phi(t, s, x, u)$ denotes the state of a system at the moment $t \in \mathbb{R}_+$, if its state at the moment $s \in \mathbb{R}_+$ was $x \in X$ and the input $u \in U_c$ was applied.

This definition is adopted from [20], but we specialize it to the systems, which satisfy classical semigroup property. Another axiomatic definitions of the control systems are also used in the literature (see [31], [37]).

In this paper we consider forward-complete and time-invariant systems. Time-invariance means, that the future evolution of a system depends only on the initial state of the system and on the applied input, but not on the initial time. For the time-invariant system we can without restriction assume that initial time $t_0 := 0$. We denote for short $\phi(t, \phi_0, u) := \phi(t, 0, \phi_0, u)$.

Definition 2 *For the formulation of stability properties the following classes of functions are useful:*

$$\begin{aligned} \mathcal{K} &:= \{\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \mid \gamma \text{ is strictly increasing, } \gamma(0) = 0\} \\ \mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\} \\ \mathcal{L} &:= \left\{ \gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \mid \gamma \text{ is continuous and decreasing with } \lim_{t \rightarrow \infty} \gamma(t) = 0 \right\} \\ \mathcal{KL} &:= \left\{ \beta : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \mid \beta(\cdot, t) \in \mathcal{K}, \forall t \geq 0, \beta(r, \cdot) \in \mathcal{L}, \forall r > 0 \right\} \end{aligned}$$

Definition 3 Σ is globally asymptotically stable at zero uniformly with respect to x (0-UGAS x), if $\exists \beta \in \mathcal{KL}$, such that $\forall \phi_0 \in X, \forall t \geq 0$ it holds

$$\|\phi(t, \phi_0, 0)\|_X \leq \beta(\|\phi_0\|_X, t). \quad (1)$$

If β can be chosen as $\beta(r, t) = Me^{-at}r \forall r, t \in \mathbb{R}_+$, for some $a, M > 0$, then Σ is called exponentially 0-UGAS x .

The notion 0-UGAS x is also called uniform asymptotic stability in the whole (see [14, p. 174]).

We need also another notion:

Definition 4 Σ is globally asymptotically stable at zero (0-GAS), if it holds

1. $\forall \varepsilon > 0 \exists \delta > 0 : \|x\|_X < \delta, t \geq 0 \Rightarrow \|\phi(t, x, 0)\|_X < \varepsilon,$
2. $\forall x \in X \|\phi(t, x, 0)\|_X \rightarrow 0, t \rightarrow \infty.$

In other words, Σ is 0-GAS if it is locally stable and globally attractive (see, e.g. [34]). Note, that the 0-UGAS x property is not equivalent to the 0-GAS in general ([14, 36], see also Section 3.1).

To study stability properties of control systems with respect to external inputs, we introduce the following notion

Definition 5 Element of state space $\phi_0 \in X$ is called an equilibrium point of control system Σ if $\phi(t, \phi_0, 0) = \phi_0$, for all $t \geq 0$.

Definition 6 Σ is called locally input-to-state stable (LISS), if $\exists \rho_x, \rho_u > 0$ and $\exists \beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that the inequality

$$\|\phi(t, \phi_0, u)\|_X \leq \beta(\|\phi_0\|_X, t) + \gamma(\|u\|_{U_c}) \quad (2)$$

holds $\forall \phi_0 : \|\phi_0\|_X \leq \rho_x, \forall t \geq 0$ and $\forall u \in U_c : \|u\|_{U_c} \leq \rho_u$.

If β can be chosen as $\beta(r, t) = Me^{-at}r \forall r, t \in \mathbb{R}_+$, for some $a, M > 0$, then Σ is called exponentially LISS (eLISS).

The control system is called *input-to-state stable (ISS)*, if in the above definition ρ_x and ρ_u can be chosen equal to ∞ .

If Σ is ISS and β can be chosen as $\beta(r, t) = Me^{-at}r \forall r, t \in \mathbb{R}_+$, for some $a, M > 0$, then Σ is called *exponentially ISS (eISS)*.

One of the most common choices for U_c is the space $U_c := PC(\mathbb{R}_+, X)$ with the norm $\|\cdot\|_{U_c} := \sup_{0 \leq s \leq \infty} \|u(s)\|_X$. In this case one can use the alternative definition of the ISS property, which is often used in the literature (see, e.g. [20], [16]):

Proposition 1 *Let $U_c := PC(\mathbb{R}_+, U)$. Then Σ is LISS iff $\exists \rho_x, \rho_u > 0$ and $\exists \beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that the inequality*

$$\|\phi(t, \phi_0, u)\|_X \leq \beta(\|\phi_0\|_X, t) + \gamma\left(\sup_{0 \leq s \leq t} \|u(s)\|_U\right) \quad (3)$$

holds $\forall \phi_0 : \|\phi_0\|_X \leq \rho_x, \forall t \geq 0$ and $\forall u \in U_c : \|u\|_{U_c} \leq \rho_u$.

Proof Sufficiency is clear, since $\sup_{0 \leq s \leq t} \|u(s)\|_U \leq \sup_{0 \leq s \leq \infty} \|u(s)\|_U = \|u\|_{U_c}$.

Now let Σ be LISS. Due to causality property of Σ the state $\phi(\tau, \phi_0, u)$, $\tau \in [0, t]$ of the system Σ does not depend on the values of $u(s)$, $s > t$. For arbitrary $t \geq 0$, $\phi_0 \in X$ and $u \in U_c$ consider another input $\tilde{u} \in U_c$, defined by

$$\tilde{u}(\tau) := \begin{cases} u(\tau), & \tau \in [0, t], \\ u(t), & \tau > t. \end{cases}$$

The inequality (2) holds for all admissible inputs, and hence it holds also for \tilde{u} . Substituting \tilde{u} into (2) and using that $\|\tilde{u}\|_{U_c} = \sup_{0 \leq s \leq t} \|u(s)\|_U$, we obtain

(3). ■

The similar property (with $\text{ess sup}_{0 \leq s \leq t} \|u(s)\|_U$ instead of $\sup_{0 \leq s \leq t} \|u(s)\|_U$) holds for continuous input functions ($U_c := C(\mathbb{R}_+, U)$), for the class of strongly measurable and essentially bounded inputs $U_c := L_\infty(\mathbb{R}_+, U)$ (which is the standard choice in the case of finite-dimensional systems) and many other classes of input functions.

3 Linear systems

Let X be a Banach space and $\mathcal{T} = \{T(t), t \geq 0\}$ be C_0 -semigroup on X with an infinitesimal generator $A = \lim_{t \rightarrow +0} \frac{1}{t}(T(t)x - x)$.

Consider a linear control system with inputs of the form

$$\begin{aligned} \dot{s} &= As + f(u(t)), \\ s(0) &= s_0, \end{aligned} \quad (4)$$

where $f : U \rightarrow X$ is continuous and so that for some $\gamma \in \mathcal{K}$ it holds

$$\|f(u)\|_X \leq \gamma(\|u\|_U), \quad \forall u \in U. \quad (5)$$

The space of input functions we take as $U_c := C([0, \infty), U)$.

We consider weak solutions of the problem (4), that is the solutions of integral equation, obtained from (4) by variation of constants formula

$$s(t) = T(t)s_0 + \int_0^t T(t-r)f(u(r))dr, \quad (6)$$

where $s_0 \in X$.

Note that in (6) the functions under the sign of integration are strongly measurable (since they are continuous, see [17, p. 84]) and for all $t \geq 0$

$$\int_0^t \|T(t-r)f(u(r))\|_X dr < \infty.$$

Thus according to the criterion of Bochner integrability (see [17, Theorem 3.7.4.]), integral in (6) is well-defined in the sense of Bochner.

The following fact is well-known

Proposition 2 *For finite-dimensional systems ($X = \mathbb{R}^n$) the following properties of the system (4) are equivalent: $e0$ -GAS, $eISS$, 0 -GAS, ISS .*

We are going to obtain a corresponding counterpart of this proposition for infinite-dimensional systems. We need the following lemma:

Lemma 1 *The following statements are equivalent:*

1. (4) is 0 -UGASx.
2. \mathcal{T} is uniformly stable (that is, $\|T(t)\| \rightarrow 0, t \rightarrow \infty$).
3. \mathcal{T} is uniformly exponentially stable ($\|T(t)\| \leq Me^{-\omega t}$ for some $M, \omega > 0$ and all $t \geq 0$).
4. (4) is exponentially 0 -UGASx.

Proof $1 \Leftrightarrow 2$. At first note that β can be always chosen as $\beta(r, t) = \zeta(t)r$ for some $\zeta \in \mathcal{L}$. Indeed, consider $x \in X : \|x\|_X = 1$, substitute it into (1) and choose $\zeta(\cdot) = \beta(1, \cdot) \in \mathcal{L}$. From linearity of \mathcal{T} we have, that $\forall x \in X, x \neq 0$ $\|T(t)x\|_X = \|x\|_X \cdot \|T(t)\frac{x}{\|x\|_X}\|_X \leq \zeta(t)\|x\|_X$.

Let (4) be 0 -UGASx. Then $\exists \zeta \in \mathcal{L}$, such that

$$\|T(t)x\|_X \leq \beta(\|x\|_X, t) = \zeta(t)\|x\|_X \quad \forall x \in X, \forall t \geq 0$$

holds. This means, that $\|T(\cdot)\| \leq \zeta(\cdot)$, and, consequently, \mathcal{T} is uniformly stable.

If \mathcal{T} is uniformly stable, then it follows, that $\exists \zeta \in \mathcal{L} : \|T(\cdot)\| \leq \zeta(\cdot)$. Then $\forall x \in X$ $\|T(t)x\|_X \leq \zeta(t)\|x\|_X$.

Equivalence $2 \Leftrightarrow 3$ is well-known (see [11, Proposition 1.2, p. 296]).

$3 \Leftrightarrow 4$. Follows from the fact that for some $M, \omega > 0$ it holds that $\|T(t)x\| \leq Me^{-\omega t}\|x\|_X \quad \forall x \in X \Leftrightarrow \|T(t)\| \leq Me^{-\omega t}$ for some $M, \omega > 0$. \blacksquare

We summarize this subsection with the following proposition

Proposition 3 *For systems of the form (4) it holds:*

$$(4) \text{ is } e0\text{-UGASx} \Leftrightarrow (4) \text{ is } 0\text{-UGASx} \Leftrightarrow (4) \text{ is } eISS \Leftrightarrow (4) \text{ is } ISS.$$

Proof System (4) is e0-UGAS $x \Leftrightarrow$ (4) 0-UGAS x by Lemma 1.

Clearly, from eISS of (4) it follows ISS of (4), and this implies that (4) is 0-UGAS x by taking $u \equiv 0$. It remains to prove, that 0-UGAS x of (4) implies eISS of (4).

Let system (4) be 0-UGAS x , then by Lemma 1, \mathcal{T} is an exponentially stable C_0 -semigroup, that is, $\exists M, w > 0$, such that $\|T(t)\| \leq Me^{-wt}$ for all $t \geq 0$. From (6) we have

$$\|s(t)\|_X \leq Me^{-wt}\|s_0\|_X + \frac{M}{w}\gamma(\|u\|_{U_c}),$$

and the eISS is proved. \blacksquare

For finite-dimensional linear systems 0-GAS is equivalent to 0-UGAS x and ISS to eISS, consequently, the Proposition 2 is a special case of Proposition 3. However, for infinite-dimensional linear systems 0-GAS and 0-UGAS x are not equivalent.

Moreover, 0-GAS in general does not imply bounded-input bounded-state (BIBS) property ($\forall x \in X, \forall u \in U_c: \|u\|_{U_c} \leq M$ for some $M > 0 \Rightarrow \|\phi(t, x, u)\|_X \leq R$ for some $R > 0$). We show this by the following example (another example, which demonstrates this property, can be found in [23]).

3.1 Counterexample

Let $C(\mathbb{R})$ be a space of continuous functions on \mathbb{R} , and let $X = C_0(\mathbb{R})$ be the Banach space of continuous functions (with sup-norm), that vanish at infinity:

$$C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) : \forall \varepsilon > 0 \exists \text{ compact set } K_\varepsilon \subset \mathbb{R} : |f(s)| < \varepsilon \forall s \in \mathbb{R} \setminus K_\varepsilon\}.$$

For a given $q \in C(\mathbb{R})$ consider the multiplication semigroup T_q (for the properties of these semigroups see, e.g., [11]), defined by

$$T_q(t)f = e^{tq}f \quad \forall f \in C_0(\mathbb{R}),$$

and for all $t \geq 0$ we define $e^{tq} : x \in \mathbb{R} \mapsto e^{tq(x)}$.

Let us take $U = X = C_0(\mathbb{R})$ and choose q as $q(s) = -\frac{1}{1+|s|}$. Consider the control system, given by

$$\dot{x} = A_q x + u, \tag{7}$$

where A_q is an infinitesimal generator of T_q .

Semigroup T_q is a strongly stable (but not exponentially stable) semigroup. Indeed, $|(T_q(t)f)(s)| = e^{-t\frac{1}{1+|s|}}|f(s)|$. But $f \in C_0(\mathbb{R})$, that is $\forall \varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \mathbb{R}$, such that $|f(s)| < \varepsilon \forall s \in \mathbb{R} \setminus K_\varepsilon$. For such ε it holds, that $|(T_q(t)f)(s)| < \varepsilon \forall s \in \mathbb{R} \setminus K_\varepsilon, \forall t \geq 0$. Moreover, there exists $t(\varepsilon)$: $|(T_q(t)f)(s)| < \varepsilon$ for all $s \in K_\varepsilon$ and $t \geq t(\varepsilon)$. Overall, we obtain, that $\|T_q(t)f\|_{C_0(\mathbb{R})} < \varepsilon \forall t \geq t(\varepsilon)$. This proves strong stability of T_q .

One can show, that the system (7) is 0-GAS.

Since $\|T_q(t)\| = 1 \ \forall t \geq 0$, there is no $\zeta \in \mathcal{L}$, such that $\|T_q(t)x\|_X \leq \zeta(t)\|x\|_X$ for all $x \in X$, and $t \geq 0$.

Take constant w.r.t. time external input $u \in C_0(\mathbb{R})$: $u(s) = a \frac{1}{\sqrt{1+|s|}}$, for some $a > 0$ and all $s \in \mathbb{R}$. The solution of (7) is given by:

$$\begin{aligned} x(t)(s) &= e^{-t \frac{1}{1+|s|}} x_0 + \int_0^t e^{-(t-r) \frac{1}{1+|s|}} \frac{a}{\sqrt{1+|s|}} dr \\ &= e^{-t \frac{1}{1+|s|}} x_0 - a \sqrt{1+|s|} (e^{-t \frac{1}{1+|s|}} - 1). \end{aligned}$$

We make a simple estimate, substituting $s = t - 1$ for $t > 1$:

$$\sup_{s \in \mathbb{R}} a \left| \sqrt{1+|s|} (e^{-t \frac{1}{1+|s|}} - 1) \right| \geq a \sqrt{t} (1 - e^{-1}) \rightarrow \infty, \quad t \rightarrow \infty.$$

For all $x_0 \in C_0(\mathbb{R})$ holds $\|e^{-t \frac{1}{1+|s|}} x_0\|_X \rightarrow 0$, $t \rightarrow \infty$. Thus, $\|x(t)\|_X \rightarrow \infty$, $t \rightarrow \infty$, and the system (7) possesses unbounded trajectories for arbitrary small inputs.

3.2 Example: linear parabolic equations with Neumann boundary conditions

In this subsection we investigate the stability of a system of parabolic equations with Neumann conditions on the boundary.

Let G be a bounded domain in \mathbb{R}^p with smooth boundary ∂G , and let Δ be Laplacian in G . Let also $F \in C(G \times \mathbb{R}^m, \mathbb{R}^n)$, $F(x, 0) \equiv 0$.

Consider a parabolic system

$$\begin{cases} \frac{\partial s(x, t)}{\partial t} - \Delta s = R s + F(x, u(x, t)), & x \in G, t > 0, \\ s(x, 0) = \phi_0(x), & x \in G, \\ \frac{\partial s}{\partial n} \Big|_{\partial G \times \mathbb{R}_+} = 0. \end{cases} \quad (8)$$

Here $\frac{\partial}{\partial n}$ is the normal derivative, $s(x, t) \in \mathbb{R}^n$, $R \in \mathbb{R}^{n \times n}$ and $u \in C(G \times \mathbb{R}_+, \mathbb{R}^m)$ be the external input.

Let $L : C(\overline{G}) \rightarrow C(\overline{G})$, $L = -\Delta$ with

$$D(L) = \{f \in C^2(G) \cap C^1(\overline{G}) : Lf \in C(\overline{G}), \frac{\partial f}{\partial n} \Big|_{\partial G} = 0\}.$$

Define the diagonal operator matrix $A = \text{diag}(-L, \dots, -L)$ with $-L$ as diagonal elements and $D(A) = (D(L))^n$. The closure \overline{A} of A is an infinitesimal generator of an analytic semigroup on $X = (C(\overline{G}))^n$.

Define a space of input values by $U := C(\overline{G}, \mathbb{R}^m)$ and the space of input functions by $U_c := C(\mathbb{R}_+, U)$.

The problem (8) may be considered as an abstract differential equation:

$$\begin{aligned} \dot{s} &= (\overline{A} + R)s + f(u(t)), \\ s(0) &= \phi_0, \end{aligned}$$

where $u \in U_c$, $u(t)(x) = u(x, t)$ and $f : U \rightarrow X$ is defined by $f(v)(x) := F(x, v(x))$.

One can check, that a map $t \rightarrow f(u(t))$ is continuous, and

$$\|f(u)\|_X = \sup_{x \in \overline{G}} |f(u)(x)| = \sup_{x \in \overline{G}} |F(x, u(x))| \leq \gamma(\|u\|_U),$$

where $\gamma(\|u\|_U) := \sup_{x \in \overline{G}, y: |y| \leq \|u\|_U} |F(x, y)|$.

Consequently we have reformulated the problem (8) in the form (4). Note that $\overline{A} + R$ also generates an analytic semigroup, as a sum of infinitesimal generator of analytic semigroup \overline{A} and bounded operator R .

Our claim is:

Proposition 4 *System (8) is eISS $\Leftrightarrow R$ is Hurwitz.*

Proof Denote by $S(t)$ the analytic semigroup, generated by $\overline{A} + R$.

We are going to find a simpler representation for $S(t)$. Consider (8) with $u \equiv 0$. Substituting in (8) $s(x, t) = e^{Rt}v(x, t)$, we obtain a simpler problem for v :

$$\begin{cases} \frac{\partial v(x, t)}{\partial t} = Av, & x \in G, t > 0, \\ v(x, 0) = \phi_0(x), & x \in G, \\ \frac{\partial v}{\partial n} \Big|_{\partial G \times \mathbb{R}_+} = 0. \end{cases} \quad (9)$$

In terms of semigroups, it means: $S(t) = e^{Rt}T(t)$, where $T(t)$ is a semigroup generated by \overline{A} . It is well-known (see, e.g. [15]), that the growth bound of analytic semigroup $T(t)$ is given by $\sup \Re(\text{Spec}(\overline{A})) = \sup_{\lambda \in \text{Spec}(\overline{A})} \Re(\lambda)$, where $\Re(z)$ is the real part of a complex number z .

We are going to find an upper bound of spectrum of \overline{A} in $D(A)$. Note, that $\text{Spec}(A) = \text{Spec}(-L)$. Thus, it is enough to estimate the spectrum of $-L$, that consists of all $\lambda \in \mathbb{C}$, such that the following equation has nontrivial solution

$$\begin{cases} Ls + \lambda s = 0, & x \in G \\ \frac{\partial s}{\partial n} \Big|_{\partial G} = 0. \end{cases} \quad (10)$$

Let $\lambda > 0$ be an eigenvalue, and $u_\lambda \neq 0$ be the corresponding eigenfunction. If u_λ attains its nonnegative maximum over \overline{G} in some $x \in G$, then, according to the strong maximum principle (see [12], p. 333), $u_\lambda \equiv \text{const}$ and consequently $u_\lambda \equiv 0 \Rightarrow u_\lambda$ cannot be an eigenfunction. If u_λ attains the non-negative maximum over \overline{G} in some $x \in \partial G$, then by Hopf's lemma (see [12], p. 330), $\frac{\partial u_\lambda(x)}{\partial n} > 0$. Consequently, $u_\lambda \leq 0$ in \overline{G} . But $-u_\lambda$ is also an eigenfunction, thus, applying the same argument, we obtain that $u_\lambda \equiv 0$ in \overline{G} , thus $\lambda > 0$ is not an eigenvalue.

Obviously $\lambda = 0$ is an eigenvalue, therefore growth bound of $T(t)$ is 0 and growth bound of $S(t)$ is $\omega_0 = \sup\{\Re(\lambda) : \exists x \neq 0 : Rx = \lambda x\}$. Thus, R to be Hurwitz is a sufficient condition for the system (8) to be exponentially 0-UGAS and, consequently, eISS.

It is also a necessary condition, because for constant ϕ_0 and $u \equiv 0$ the solutions of (8) are for arbitrary $x \in G$ the solutions of $\dot{s} = Rs$, and to guarantee the stability of these solutions R has to be Hurwitz. \blacksquare

In (8) the diffusion coefficients are equal to one. In case, when the diffusion coefficients of different subsystems are not equal to each other the statement of Proposition 4 is in general not true because of Turing instability phenomenon (see [35], [25]).

4 Lyapunov functions for nonlinear systems

To verify both local and global input-to-state stability of nonlinear systems, Lyapunov functions can be exploited. In this section we provide basic tools and illustrate them by an example.

Definition 7 A continuous function $V : D \mapsto \mathbb{R}_+$, $D \subset X$, $0 \in \text{int}(D) = D \setminus \partial D$ is called local ISS-Lyapunov function (LISS-LF) for Σ , if $\exists \rho_x, \rho_u > 0$ and functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$ and positive definite function α , such that:

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \in D$$

and $\forall x \in X : \|x\|_X \leq \rho_x$, $\forall u \in U_c : \|u\|_{U_c} \leq \rho_u$ it holds:

$$\|x\|_X \geq \chi(\|u\|_{U_c}) \Rightarrow \dot{V}_u(x) \leq -\alpha(\|x\|_X), \quad (11)$$

where the Lie derivative of V corresponding to the input u is given by

$$\dot{V}_u(x) = \overline{\lim}_{t \rightarrow +0} \frac{1}{t} (V(\phi(t, x, u)) - V(x)). \quad (12)$$

Function χ is called ISS-Lyapunov gain for (X, U_c, ϕ) .

If in the previous definition $D = X$, $\rho_x = \infty$ and $\rho_u = \infty$, then the function is called ISS-Lyapunov function.

Note, that in general a computation of the Lie derivative $\dot{V}_u(x)$ requires knowledge of the input on some neighborhood of the time instant $t = 0$.

If the input, with respect to which the Lie derivative $\dot{V}_u(x)$ is computed, is clear from the context, then we write simply $\dot{V}(x)$.

Theorem 1 Let $\Sigma = (X, U_c, \phi)$ be a time-invariant control system, and $x \equiv 0$ be its equilibrium point.

Also let for all $u \in U_c$ and for all $s \geq 0$ a function \tilde{u} , defined by $\tilde{u}(\tau) = u(\tau + s)$ for all $\tau \geq 0$, belong to U_c and $\|\tilde{u}\|_{U_c} \leq \|u\|_{U_c}$.

If Σ possesses a LISS-Lyapunov function, then it is LISS.

For a counterpart of this theorem for infinite-dimensional dynamical systems (without inputs) see, e.g., [15].

Proof Let the control system $\Sigma = (X, U_c, \phi)$ possess a LISS-Lyapunov function and $\psi_1, \psi_2, \chi, \alpha, \rho_x, \rho_u$ be as in the Definition 7. Take an arbitrary control $u \in U_c$ with $\|u\|_{U_c} \leq \rho_u$ such that

$$I = \{x \in D : \|x\|_X \leq \rho_x, V(x) \leq \psi_2 \circ \chi(\|u\|_{U_c}) \leq \rho_x\} \subset \text{int}(D).$$

Such u exists, because $0 \in \text{int}(D)$.

Firstly we prove, that I is invariant w.r.t. Σ , that is: $\forall x \in I \Rightarrow x(t) = \phi(t, x, u) \in I, t \geq 0$.

If $u \equiv 0$, then $I = \{0\}$, and I is invariant, because $x = 0$ is the equilibrium point of Σ . Consider $u \neq 0$.

If I is not invariant w.r.t. Σ , then, due to continuity of ϕ w.r.t. t (Continuity axiom of Σ), $\exists t_* > 0$, such that $V(x(t_*)) = \psi_2 \circ \chi(\|u\|_{U_c})$, and therefore $\|x(t_*)\|_X \geq \chi(\|u\|_{U_c})$.

The input to the system Σ after time t^* is \tilde{u} , defined by $\tilde{u}(\tau) = u(\tau + t^*)$, $\tau \geq 0$. According to the assumptions of the theorem $\|\tilde{u}\|_{U_c} \leq \|u\|_{U_c}$. Then from (11) it follows, that $\dot{V}_{\tilde{u}}(x(t_*)) = -\alpha(\|x(t_*)\|_X) < 0$. Thus, the trajectory cannot escape the set I .

Now take arbitrary x_0 : $\|x_0\|_X \leq \rho_x$. As long as $x_0 \notin I$, we have the following differential inequality ($x(t)$ is the trajectory, corresponding to the initial condition x_0):

$$\dot{V}(x(t)) \leq -\alpha(\|x(t)\|_X) \leq -\alpha \circ \psi_2^{-1}(V(x(t))).$$

From the comparison principle (see [22], Lemma 4.4 for $y(t) = V(x(t))$) it follows, that $\exists \tilde{\beta} \in \mathcal{KL}$: $V(x(t)) \leq \tilde{\beta}(V(x_0), t)$, and consequently:

$$\|x(t)\|_X \leq \beta(\|x_0\|_X, t), \forall t : x(t) \notin I, \quad (13)$$

where $\beta(r, t) = \psi_1^{-1} \circ \tilde{\beta}(\psi_2^{-1}(r), t)$, $\forall r, t \geq 0$.

From the properties of \mathcal{KL} functions it follows, that $\exists t_1$:

$$t_1 := \inf_{t \geq 0} \{x(t) = \phi(t, x_0, u) \in I\}.$$

From the invariance of the set I we conclude, that

$$\|x(t)\|_X \leq \gamma(\|u\|_{U_c}), \quad t > t_1, \quad (14)$$

where $\gamma = \psi_1^{-1} \circ \psi_2 \circ \chi \in \mathcal{K}$.

Our estimates hold for arbitrary control u : $\|u\|_{U_c} \leq \rho_u$, thus, combining (13) and (14), we obtain the claim of the theorem. \blacksquare

Remark 1 As a special case we have, that if under the same assumptions a control system possesses an ISS-Lyapunov function, then it is ISS.

Remark 2 Assumption on the properties of U_c used in the Theorem 1 holds for many usual function classes, such as $PC(\mathbb{R}_+, U)$, $L_p(\mathbb{R}_+, U)$, $p \geq 1$, $L_\infty(\mathbb{R}_+, U)$, Sobolev spaces etc.

The Definition 7 differs from the definition of (L)ISS-Lyapunov function, used in finite-dimensional theory (see, e.g. [33]). We are going to prove, that for the ODE systems our definition is equivalent to the standard one.

Firstly we reformulate the definition of LISS-LF for the case, when $U_c = PC(\mathbb{R}_+, U)$.

Proposition 5 *Continuous function $V : D \rightarrow \mathbb{R}_+$, $D \subset X$, $0 \in \text{int}(D) = D \setminus \partial D$ is a LISS-Lyapunov function for $\Sigma = (X, PC(\mathbb{R}_+, U), \phi)$ if and only if there exist $\rho_x, \rho_u > 0$ and functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\tilde{\chi} \in \mathcal{K}$ and positive definite function α , such that:*

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \in D$$

and $\forall x \in X : \|x\|_X \leq \rho_x, \forall \xi \in U : \|\xi\|_U \leq \rho_u$ it holds

$$\|x\|_X \geq \tilde{\chi}(\|\xi\|_U) \Rightarrow \dot{V}_u(x) \leq -\alpha(\|x\|_X), \quad (15)$$

for all $u \in U_c : \|u\|_{U_c} \leq \rho_u$ with $u(0) = \xi$.

Proof Let us begin with sufficiency. Let $u \in U_c = PC(\mathbb{R}_+, U)$, $\|u\|_{U_c} \leq \rho_u$. Take arbitrary $x \in X$ and assume that $\|x\|_X \geq \chi(\|u\|_{U_c})$. Then $\|x\|_X \geq \chi(\|u(0)\|_U)$ and according to (15) for this u it holds $\dot{V}_u(x) \leq -\alpha(\|x\|_X)$. The implication (11) is proved and thus V is a LISS-Lyapunov function according to Definition 7.

Let us prove necessity. Take arbitrary $u \in U_c$, and for arbitrary $s > 0$ consider the input $u_s \in U_c$ defined by

$$u_s(\tau) := \begin{cases} u(\tau), \tau \in [0, s], \\ u(s), \tau > s. \end{cases}$$

Due to Causality of Σ , $\phi(t, x, u) = \phi(t, x, u_s)$ for all $t \in [0, s]$, and according to the definition of Lie derivative we obtain $\dot{V}_u(x) = \dot{V}_{u_s}(x)$. Let $u \in U_c$ and $\|u\|_{U_c} \leq \rho_u$. Then also $\|u_s\|_{U_c} \leq \rho_u$ and since V is a LISS-Lyapunov function it follows that

$$\|x\|_X \geq \chi(\|u_s\|_{U_c}) \Rightarrow \dot{V}_{u_s}(x) \leq -\alpha(\|x\|_X).$$

Then it holds also

$$\|x\|_X \geq \chi(\|u_s\|_{U_c}) \Rightarrow \dot{V}_u(x) \leq -\alpha(\|x\|_X). \quad (16)$$

Since $U_c = PC(\mathbb{R}_+, U)$, it follows, that for arbitrary $u \in U_c$ and arbitrary $\varepsilon > 0$ there exists $\tau > 0$ such that $\|u_\tau\|_{U_c} \leq (1 + \varepsilon)\|u(0)\|_U$. Then from (16) it follows, that

$$\|x\|_X \geq \tilde{\chi}(\|u(0)\|_U) \Rightarrow \dot{V}_u(x) \leq -\alpha(\|x\|_X),$$

where $\tilde{\chi}(r) = \chi((1 + \varepsilon)r)$, for all $r \geq 0$.

Since $u \in U_c$, $\|u\|_{U_c} \leq \rho_u$ has been chosen arbitrarily, the necessity is proved. \blacksquare

Now consider the ODE system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m. \quad (17)$$

System (17) defines a time-invariant control system $\Sigma = (X, U_c, \phi)$, where $X = \mathbb{R}^n$, $U_c = L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ and $\phi(t, x_0, u)$ is a solution of (17) subject to a given input $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ and initial condition $x(0) = x_0$.

Let $V : D \rightarrow \mathbb{R}_+$, $D \subset \mathbb{R}^n$, $0 \in \text{int}(D) = D \setminus \partial D$ be locally Lipschitz continuous function (and thus it is differentiable almost everywhere by Rademacher's theorem). For such systems $\dot{V}_u(x)$ can be computed for almost all x and the implication (15) resolves to

$$\|x\|_X \geq \chi(\|\xi\|_U) \Rightarrow \nabla V \cdot f(x, \xi) \leq -\alpha(\|x\|_X).$$

Using this implication instead of (15), we obtain the standard definition of LISS-Lyapunov function for finite-dimensional systems. Thus, Definition 7 is consistent with the existing definitions of LISS-Lyapunov functions for ODE systems.

Note that the system (17) is time-invariant, for the space $L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ the assumption of the Theorem 1 holds, and we obtain basic result from finite-dimensional theory that existence of an (L)ISS-Lyapunov function implies its (L)ISS.

4.1 Example

Let us consider the following system

$$\begin{cases} \frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial x^2} - f(s) + u^m(x, t), & x \in (0, \pi), t > 0, \\ s(0, t) = s(\pi, t) = 0. \end{cases} \quad (18)$$

We assume, that f is locally Lipschitz continuous, monotonically increasing up to infinity, $f(-r) = -f(r)$ for all $r \in \mathbb{R}$ (in particular, $f(0) = 0$), and $m \in (0, 1]$.

To reformulate (18) as an abstract differential equation we define $As = \frac{d^2 s}{dx^2}$ with $D(A) = H_0^1(0, \pi) \cap H^2(0, \pi)$.

The norm on $H_0^1(0, \pi)$ we define by

$$\|s\|_{H_0^1(0, \pi)} = \left(\int_0^\pi \left(\frac{\partial s}{\partial x} \right)^2 dx \right)^{\frac{1}{2}}.$$

Operator A generates an analytic semigroup on $L_2(0, \pi)$. System (18) takes the form

$$\frac{\partial s}{\partial t} = As - f(s) + u^m, \quad t > 0. \quad (19)$$

Equation (19) defines a control system with state space $X = H_0^1(0, \pi)$ and input function space $U_c = C(\mathbb{R}_+, L_2(0, \pi))$.

Consider the following ISS-Lyapunov function candidate:

$$V(s) = \int_0^\pi \left(\frac{1}{2} \left(\frac{\partial s}{\partial x} \right)^2 + \int_0^{s(x)} f(y) dy \right) dx. \quad (20)$$

We are going to prove, that V is an ISS-Lyapunov function.

Under above assumptions about function f it holds, that $\int_0^r f(y)dy \geq 0$ for every $r \in \mathbb{R}$. Thus, V is positive definite:

$$V(s) \geq \int_0^\pi \frac{1}{2} \left(\frac{\partial s}{\partial x} \right)^2 dx = \frac{1}{2} \|s\|_{H_0^1(0,\pi)}^2. \quad (21)$$

Let us compute the Lie derivative of V

$$\begin{aligned} \dot{V}(s) &= \int_0^\pi \frac{\partial s}{\partial x} \frac{\partial^2 s}{\partial x \partial t} + f(s(x)) \frac{\partial s}{\partial t} dx \\ &= \left[\frac{\partial s}{\partial x} \frac{\partial s}{\partial t} \right]_{x=0}^{x=\pi} + \int_0^\pi \left(-\frac{\partial^2 s}{\partial x^2} \frac{\partial s}{\partial t} + f(s(x)) \frac{\partial s}{\partial t} \right) dx. \end{aligned}$$

From boundary conditions it follows $\frac{\partial s}{\partial t}(0, t) = \frac{\partial s}{\partial t}(\pi, t) = 0$. Thus, substituting expression for $\frac{\partial s}{\partial t}$, we obtain

$$\dot{V}(s) = - \int_0^\pi \left(\frac{\partial^2 s}{\partial x^2} - f(s(x)) \right)^2 dx + \int_0^\pi \left(\frac{\partial^2 s}{\partial x^2} - f(s(x)) \right) (-u^m) dx.$$

Define

$$I(s) := \int_0^\pi \left(\frac{\partial^2 s}{\partial x^2} - f(s(x)) \right)^2 dx.$$

Using Cauchy-Schwarz inequality for the second term, we have:

$$\dot{V}(s) \leq -I(s) + \sqrt{I(s)} \|u^m\|_{L_2(0,\pi)}. \quad (22)$$

Now let us consider $I(s)$

$$\begin{aligned} I(s) &= \int_0^\pi \left(\frac{\partial^2 s}{\partial x^2} \right)^2 dx - 2 \int_0^\pi \frac{\partial^2 s}{\partial x^2} f(s(x)) dx + \int_0^\pi f^2(s(x)) dx \\ &= \int_0^\pi \left(\frac{\partial^2 s}{\partial x^2} \right)^2 dx + 2 \int_0^\pi \left(\frac{\partial s}{\partial x} \right)^2 \frac{\partial f}{\partial s}(s(x)) dx + \int_0^\pi f^2(s(x)) dx \\ &\geq \int_0^\pi \left(\frac{\partial^2 s}{\partial x^2} \right)^2 dx. \end{aligned}$$

For $s \in H_0^1(0, \pi) \cap H^2(0, \pi)$ it holds (see [15], p. 85), that

$$\int_0^\pi \left(\frac{\partial^2 s}{\partial x^2} \right)^2 dx \geq \int_0^\pi \left(\frac{\partial s}{\partial x} \right)^2 dx.$$

Overall, we have:

$$I(s) \geq \|s\|_{H_0^1(0,\pi)}^2. \quad (23)$$

Let us consider $\|u^m\|_{L_2(0,\pi)}$. Using Hölder inequality, we obtain:

$$\|u^m\|_{L_2(0,\pi)} = \left(\int_0^\pi u^{2m} \cdot 1 dx \right)^{\frac{1}{2}} \quad (24)$$

$$\leq \left(\int_0^\pi u^2 dx \right)^{\frac{m}{2}} \left(\int_0^\pi 1^{\frac{1}{1-m}} dx \right)^{\frac{1-m}{2}} = \pi^{\frac{1-m}{2}} \|u\|_{L_2(0,\pi)}^m. \quad (25)$$

Now we choose the gain as

$$\chi(r) = a\pi^{\frac{1-m}{2}} r^m, \quad a > 1.$$

If $\chi(\|u\|_{L_2(0,\pi)}) \leq \|s\|_{H_0^1(0,\pi)}$, we obtain from (22), using (24) and (23):

$$\dot{V}(s) \leq -I(s) + \frac{1}{a} \sqrt{I(s)} \|s\|_{H_0^1(0,\pi)} \leq \left(\frac{1}{a} - 1\right) I(s) \leq \left(\frac{1}{a} - 1\right) \|s\|_{H_0^1(0,\pi)}^2. \quad (26)$$

Note that although some of the calculations above hold only for smooth functions, but for every $s \in H_0^1(0,\pi)$ there exists a sequence of the functions $\{s_m\} \in C^\infty([0,\pi])$, $s_m(0) = s_m(\pi) = 0$ so that $s_m \rightarrow s$ in $H_0^1(0,\pi)$ (see, e.g. [12, p. 252]). From this fact one can prove, that the estimation (26) holds for all $s \in H_0^1(0,\pi)$.

We have proved, that (20) is an ISS-Lyapunov function, and consequently, (19) (with $X = H_0^1(0,\pi)$, $U_c = C(\mathbb{R}_+, L_2(0,\pi))$) is ISS.

Remark 3 In the example we have taken $U = L_2(0,\pi)$ and $X = H_0^1(0,\pi)$. But in case of interconnection with other parabolic systems (when we identify input u with the state of the other system), that have state space $H_0^1(0,\pi)$ (as our system), we have to choose $U = X = H_0^1(0,\pi)$. In this case we can continue estimates (24), using Friedrichs' inequality

$$\int_0^\pi s^2(x) dx \leq \int_0^\pi \left(\frac{\partial s}{\partial x} \right)^2 dx$$

to obtain

$$\|u^m\|_{L_2(0,\pi)} \leq \pi^{\frac{1-m}{2}} \|u\|_{H_0^1(0,\pi)}^m \quad (27)$$

and choosing the same gains, prove the input-to state stability of (19) w.r.t. spaces $X = H_0^1(0,\pi)$, $U_c = C(\mathbb{R}_+, H_0^1(0,\pi))$.

Remark 4 The input-to-state stability for semilinear parabolic PDEs has been studied also in the recent paper [23]. However, the definition of ISS and of ISS-Lyapunov function in that paper are different from used in our paper.

5 Linearization

In this section we prove a linearization principle that allows us to investigate the local ISS of a system using solely information about ISS of the corresponding linearized system.

We assume that X is a Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$, and A generates an analytic semigroup on X . Consider a system

$$\dot{x} = Ax + f(x(t), u(t)), \quad x(t) \in X, u(t) \in U, \quad (28)$$

where function $f : X \times U \rightarrow X$ is defined on some open set Q , $(0,0) \in Q$.

Also we suppose that $u : \mathbb{R}_+ \rightarrow U$ is Hölder-continuous and f is Lipschitz continuous with respect to the first and Hölder continuous with respect to the

second argument. That is, if $(x, u) \in Q$, then there exists a neighborhood W of (x, u) : $W \subset Q$, such that $\forall (y, v) \in W$ it holds

$$\|f(x, u) - f(y, v)\|_X \leq L(\|u - v\|_U^\theta + \|x - y\|_X),$$

for some constants $L, \theta > 0$.

We assume that $f(0, 0) = 0$, thus $x \equiv 0$ is an equilibrium point of (28).

Theorem 2 *Let in (28)*

$$f(x, u) = Bx + Cu + g(x, u),$$

where $B \in L(X)$ and $C \in L(U, X)$ and for each constant $w > 0 \exists \rho > 0$, such that $\forall x : \|x\|_X \leq \rho, \forall u : \|u\|_U \leq \rho$ it holds

$$\|g(x, u)\|_X \leq w(\|x\|_X + \|u\|_U).$$

If the system

$$\dot{x} = Ax + Bx + Cu \tag{29}$$

is eISS, then (28) is LISS.

Proof Operator A is an infinitesimal generator of an analytic semigroup and B is bounded, therefore $R = A + B$ also generates an analytic semigroup.

System (29) is eISS and consequently exponentially 0-UGAS, therefore there exists (see, e.g., [3]) a Lyapunov function for (29)

$$V(x) = \langle Px, x \rangle, \tag{30}$$

where $P \in L(X)$ is a positive bounded operator, for which it holds that

$$\langle Rx, Px \rangle + \langle Px, Rx \rangle = -\|x\|_X^2, \quad \forall x \in D(R). \tag{31}$$

We are going to prove that V is a LISS-Lyapunov function for the system (28) for properly chosen gains. Let us compute Lie derivative of V w.r.t. the system (28).

Firstly consider the case, when $x \in D(R) = D(A)$.

$$\begin{aligned} \dot{V}(x) &= \langle P\dot{x}, x \rangle + \langle Px, \dot{x} \rangle \\ &= \langle P(Rx + Cu + g(x, u)), x \rangle + \langle Px, Rx + Cu + g(x, u) \rangle \\ &= \langle P(Rx), x \rangle + \langle Px, Rx \rangle + \langle P(Cu + g(x, u)), x \rangle + \langle Px, Cu + g(x, u) \rangle. \end{aligned}$$

We continue estimates using the property

$$\langle P(Rx), x \rangle = \langle Rx, Px \rangle,$$

which holds for positive operators, equality (31) and for the last two terms Cauchy-Schwarz inequality in space X

$$\begin{aligned} \dot{V}(x) &\leq -\|x\|_X^2 + \|P(Cu + g(x, u))\|_X \|x\|_X + \|Px\|_X \|Cu + g(x, u)\|_X \\ &\leq -\|x\|_X^2 + \|P\| \|(Cu + g(x, u))\|_X \|x\|_X + \|P\| \|x\|_X \|Cu + g(x, u)\|_X \\ &\leq -\|x\|_X^2 + 2\|P\| \|x\|_X (\|C\| \|u\|_U + \|g(x, u)\|_X) \end{aligned}$$

For each $w > 0 \exists \rho$, such that $\forall x : \|x\|_X \leq \rho, \forall u : \|u\|_U \leq \rho$ it holds

$$\|g(x, u)\|_X \leq w(\|x\|_X + \|u\|_U).$$

Using this inequality we continue estimates

$$\dot{V}(x) \leq -\|x\|_X^2 + 2w\|P\|\|x\|_X^2 + 2\|P\|(\|C\| + w)\|x\|_X\|u\|_U$$

Take $\chi(r) := \sqrt{r}$. Then for $\|u\|_U \leq \chi^{-1}(\|x\|_X) = \|x\|_X^2$ we have:

$$\dot{V}(x) \leq -\|x\|_X^2 + 2w\|P\|\|x\|_X^2 + 2\|P\|(\|C\| + w)\|x\|_X^3. \quad (32)$$

Choosing w and ρ small enough the right hand side can be estimated from above by some negative quadratic function of $\|x\|_X$.

These derivations hold for $x \in D(R) \subset X$. If $x \notin D(R)$, then for all admissible u the solution $x(t) \in D(R)$ and $t \rightarrow V(x(t))$ is a continuous differentiable function for all $t > 0$ (these properties follow from the properties of solutions $x(t)$, see Theorem 3.3.3 in [15]).

Therefore, by the mean-value theorem, $\forall t > 0 \exists t_* \in (0, t)$

$$\frac{1}{t}(V(x(t)) - V(x)) = \dot{V}(x(t_*)).$$

Taking the limit when $t \rightarrow +0$ we obtain that (32) holds for all $x \in X$.

This proves that V is a LISS-Lyapunov function with $\|x\|_X \leq \rho, \|u\|_U \leq \rho$ and consequently (28) is LISS. \blacksquare

In particular, the theorem holds for finite dimensional systems. In this case the assumptions about Hölder continuity for functions u and for f w.r.t. second argument (needed to achieve existence and uniqueness of solutions of (28) (see Section 3.3 in [15] for $\alpha = 0$)) can be replaced with milder assumptions (see [1], p. 183).

6 Interconnections of input-to-state stable systems

In this section we study input-to-state stability of an interconnection of n ISS systems and provide a generalization of Lyapunov small-gain theorem from [7] for the case of infinite-dimensional systems.

Consider the interconnected systems of the following form

$$\begin{cases} \dot{x}_i = A_i x_i + f_i(x_1, \dots, x_n, u), & x_i(t) \in X_i, u(t) \in U \\ i = 1, \dots, n, \end{cases} \quad (33)$$

where the state space of i -th subsystem X_i is a Banach space and A_i is a generator of C_0 - semigroup on X_i , $i = 1, \dots, n$. The space U_c we take as $U_c = PC(\mathbb{R}_+, U)$ for some Banach space of input values U .

The state space of the system (33) we denote by $X = X_1 \times \dots \times X_n$, which is Banach with the norm $\|\cdot\|_X := \|\cdot\|_{X_1} + \dots + \|\cdot\|_{X_n}$.

The input space for the i -th subsystem is $\tilde{X}_i := X_1 \times \dots \times X_{i-1} \times X_{i+1} \times \dots \times X_n \times U$. The norm in \tilde{X}_i is given by

$$\|\cdot\|_{\tilde{X}_i} := \|\cdot\|_{X_1} + \dots + \|\cdot\|_{X_{i-1}} + \|\cdot\|_{X_{i+1}} + \dots + \|\cdot\|_{X_n} + \|\cdot\|_U.$$

The elements of \tilde{X}_i we denote by $\tilde{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, u) \in \tilde{X}_i$.

The transition map of the i -th subsystem we denote by $\phi_i : \mathbb{R}_+ \times X_i \times PC(\mathbb{R}_+, \tilde{X}_i) \rightarrow X_i$. Define

$$x = (x_1, \dots, x_n), \quad f(x, u) = (f_1(x, u), \dots, f_n(x, u)), \quad A = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{pmatrix}$$

where $x_i \in X_i$, $i = 1, \dots, n$. Domain of definition of A is given by $D(A) = D(A_1) \times \dots \times D(A_n)$. Clearly A is a generator of C_0 -semigroup on X .

We rewrite the system (33) in the vector form:

$$\dot{x} = Ax + f(x, u). \quad (34)$$

Since the inputs are piecewise continuous functions, then according to Proposition 5 a function $V_i : X_i \rightarrow \mathbb{R}_+$ is an ISS-Lyapunov function for the i -th subsystem, if there exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$ and positive definite function α_i , such that

$$\psi_{i1}(\|x_i\|_{X_i}) \leq V_i(x_i) \leq \psi_{i2}(\|x_i\|_{X_i}), \quad \forall x_i \in X_i$$

and $\forall x_i \in X_i$, $\forall \tilde{x}_i \in \tilde{X}_i$, for all $v \in PC(\mathbb{R}_+, \tilde{X}_i)$ with $v(0) = \tilde{x}_i$ it holds the implication

$$\|x_i\|_{X_i} \geq \chi(\|\tilde{x}_i\|_{\tilde{X}_i}) \Rightarrow \dot{V}_i(x_i) \leq -\alpha_i(V_i(x_i)), \quad (35)$$

where

$$\dot{V}_i(x_i) = \overline{\lim}_{t \rightarrow +0} \frac{1}{t} (V_i(\phi_i(t, x_i, v)) - V_i(x_i)).$$

We are going to rewrite the implication (35) in a more suitable form. We have

$$\begin{aligned} \psi_{i1}^{-1}(V_i(x_i)) &\geq \|x_i\|_{X_i} \geq \chi(\|\tilde{x}_i\|_{\tilde{X}_i}) = \chi\left(\sum_{j=1, j \neq i}^n \|x_j\|_{X_j} + \|u\|_U\right) \\ &\geq \frac{1}{n+1} \max\left\{\max_{j=1, j \neq i}^n \{\chi(\|x_j\|_{X_j})\}, \chi(\|u\|_U)\right\} \end{aligned}$$

Therefore if $\|x_i\|_{X_i} \geq \chi(\|\tilde{x}_i\|_{\tilde{X}_i})$ holds, then also

$$V_i(x_i) \geq \max\left\{\max_{j=1}^n \chi_{ij}(V_j(x_j)), \chi_i(\|u\|_U)\right\}$$

holds with

$$\chi_{ij}(r) := \psi_{i1} \left(\frac{1}{n+1} \chi(\psi_{i2}^{-1}(r)) \right), \quad \chi_i(r) := \psi_{i1} \left(\frac{1}{n+1} \chi(r) \right), \quad i \neq j, \quad r \geq 0.$$

And thus if (35) holds, then it holds also the implication

$$V_i(x_i) \geq \max_{j=1}^n \{ \chi_{ij}(V_j(x_j)), \chi_i(\|u\|_U) \} \Rightarrow \dot{V}_i(x_i) \leq -\alpha_i(V_i(x_i)). \quad (36)$$

The statement, that if (36) holds, then so is (35) can be checked in the same way.

Remark 5 Note, that we have used in our derivations the certain norm on the space \tilde{X}_i . For finite-dimensional \tilde{X}_i such derivations can be made for arbitrary norm in \tilde{X}_i due to equivalence of the norms in a finite-dimensional space. However, for infinite-dimensional systems it is not always true.

In the following we will use the implication form as in (36) and assume, that for all $i = 1, \dots, n$ for Lyapunov function V_i of the i -th system the gains χ_{ij} , $j = 1, \dots, n$ and χ_i are given.

Gains χ_{ij} characterize the interconnection structure of subsystems. Let us introduce the gain operator $\Gamma : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ defined by

$$\Gamma(s) := \left(\max_{j=1}^n \chi_{1j}(s_j), \dots, \max_{j=1}^n \chi_{nj}(s_j) \right), \quad s \in \mathbb{R}_+^n. \quad (37)$$

For arbitrary $x, y \in \mathbb{R}^n$ define the relations " \geq " and " $<$ " on \mathbb{R}^n by

$$x \geq y \Leftrightarrow x_i \geq y_i \quad \forall i = 1, \dots, n,$$

$$x < y \Leftrightarrow x_i < y_i \quad \forall i = 1, \dots, n.$$

We recall the notion of Ω -path (see [10, 29]), useful for investigation of stability of interconnected systems and for construction of a Lyapunov function of the whole interconnection.

Definition 8 A function $\sigma = (\sigma_1, \dots, \sigma_n)^T : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$, where $\sigma_i \in \mathcal{K}_\infty$, $i = 1, \dots, n$ is called an Ω -path, if it possesses the following properties:

1. σ_i^{-1} is locally Lipschitz continuous on $(0, \infty)$;
2. for every compact set $P \subset (0, \infty)$ there are finite constants $0 < K_1 < K_2$ such that for all points of differentiability of σ_i^{-1} we have

$$0 < K_1 \leq (\sigma_i^{-1})'(r) \leq K_2, \quad \forall r \in P;$$

3.

$$\Gamma(\sigma(r)) < \sigma(r), \quad \forall r > 0. \quad (38)$$

If operator Γ satisfies the small-gain condition, namely for all $\forall s \in \mathbb{R}_+^n \setminus \{0\}$ it holds

$$\Gamma(s) \not\geq s \Leftrightarrow \exists i : (\Gamma(s))_i < s_i, \quad (39)$$

then Ω -path can be constructed as follows (see [20], Proposition 2.7 and Remark 2.8):

$$\sigma(t) = Q(at), \forall t \geq 0, \text{ for some } a \in \text{int}(\mathbb{R}_+^n), \quad (40)$$

where $Q : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ is defined by

$$Q(x) = \text{MAX}\{x, \Gamma(x), \Gamma^2(x), \dots, \Gamma^{n-1}(x)\},$$

with $\Gamma^n(x) = \Gamma \circ \Gamma^{n-1}(x)$, for all $n \geq 2$. The function MAX for all $u_i \in \mathbb{R}^n$, $i = 1, \dots, m$ is defined by

$$z = \text{MAX}\{u_1, \dots, u_m\} \in \mathbb{R}^n, \quad z_i = \max\{u_{1i}, \dots, u_{mi}\}.$$

Note that Ω -path (40) is only Lipschitz continuous, but with the help of standard mollification arguments (see, [13], Appendix B.2 or [28], Lemma 1.1.6) it can be made smooth.

Now we can state a theorem, that provides sufficient conditions for a network, consisting of n ISS subsystems to be ISS.

Theorem 3 *Let for each subsystem of (33) V_i be the ISS-Lyapunov function with corresponding gains χ_{ij} . If the corresponding operator Γ defined by (37) satisfies the small-gain condition (39), then the whole system (34) is ISS and possesses ISS-Lyapunov function defined by*

$$V(x) := \max_i \{\sigma_i^{-1}(V_i(x_i))\}, \quad (41)$$

where $\sigma = (\sigma_1, \dots, \sigma_n)^T$ is an Ω -path. The Lyapunov gain of the whole system is

$$\chi(r) := \max_i \sigma_i^{-1}(\chi_i(r)).$$

For the proof we use the following standard fact from analysis

Lemma 2 *Let $f_i : \mathbb{R} \rightarrow \mathbb{R}$ are defined and bounded in some neighborhood D of $t = 0$. Then it holds*

$$\overline{\lim}_{t \rightarrow 0} \max_{1 \leq i \leq m} \{f_i(t)\} = \max_{1 \leq i \leq m} \{\overline{\lim}_{t \rightarrow 0} f_i(t)\} \quad (42)$$

The idea of the proof is taken from [7].

Proof In order to prove that V is a Lyapunov function it is suitable to divide its domain of definition into subsets on which V takes a simpler form. Thus, for all $i \in \{1, \dots, n\}$ define a set

$$M_i = \{x \in X : \sigma_i^{-1}(V_i(x_i)) > \sigma_j^{-1}(V_j(x_j)), \forall j = 1, \dots, n, j \neq i\}.$$

From the continuity of V_i and σ_i^{-1} , $i = 1, \dots, n$ it follows that all M_i are open. Also note that $X = \cup_{i=1}^n \overline{M}_i$ and for all $i \neq j$ holds $M_i \cap M_j = \emptyset$. Define

$$\gamma(r) := \max_{j=1}^n \sigma_j^{-1} \circ \gamma_j(r).$$

Take some $i \in \{1, \dots, n\}$ and pick any $x \in M_i$. Assume that $V(x) \geq \gamma(\|u\|_U)$ holds. Then we obtain

$$\sigma_i^{-1}(V_i(x_i)) = V(x) \geq \gamma(\|u\|_U) = \max_{j=1}^n \sigma_j^{-1} \circ \gamma_j(\|u\|_U) \geq \sigma_i^{-1}(\gamma_i(\|u\|_U)).$$

But $\sigma_i^{-1} \in \mathcal{K}_\infty$, hence it holds

$$V_i(x_i) \geq \gamma_i(\|u\|_U). \quad (43)$$

On the other hand, from the condition (38) we obtain that

$$\begin{aligned} V_i(x_i) = \sigma_i(V(x)) &\geq \max_{j=1}^n \chi_{ij}(\sigma_j(V(x))) = \max_{j=1}^n \chi_{ij}(\sigma_j(\sigma_i^{-1}(V_i(x_i)))) \\ &> \max_{j=1}^n \chi_{ij}(\sigma_j(\sigma_j^{-1}(V_j(x_j)))) = \max_{j=1}^n \chi_{ij}(V_j(x_j)). \end{aligned}$$

Combining with (43) we obtain

$$V_i(x_i) \geq \max \left\{ \max_{j=1}^n \chi_{ij}(V_j(x_j)), \gamma_i(\|u\|_U) \right\} \quad (44)$$

Hence condition (36) implies that for all x the following estimate holds

$$\begin{aligned} \frac{d}{dt}V(x) &= \frac{d}{dt}(\sigma_i^{-1}(V_i(x_i))) = (\sigma_i^{-1})'(V_i(x_i)) \frac{d}{dt}V_i(x_i(t)) \\ &\leq -(\sigma_i^{-1})'(V_i(x_i))\alpha_i(V_i(x_i)) = -(\sigma_i^{-1})'(\sigma_i(V(x)))\alpha_i(\sigma_i(V(x))). \end{aligned}$$

We set

$$\alpha(r) := \min_{i=1}^n \left\{ (\sigma_i^{-1})'(\sigma_i(r))\alpha_i(\sigma_i(r)) \right\}.$$

Function α is positive definite, because $\sigma_i^{-1} \in \mathcal{K}_\infty$ and all α_i are positive definite functions. Overall, for all $x \in \cup_{i=1}^n M_i$ holds

$$\frac{d}{dt}V(x) \leq -\min_{i=1}^n (\sigma_i^{-1})'(\sigma_i(V(x)))\alpha_i(\sigma_i(V(x))) = -\alpha(V(x)).$$

Now let $x \notin \cup_{i=1}^n M_i$. From $X = \cup_{i=1}^n \overline{M}_i$ it follows that $x \in \cap_{i \in I(x)} \partial M_i$ for some index set $I(x) \subset \{1, \dots, n\}$, $|I(x)| \geq 2$.

$$\cap_{i \in I(x)} \partial M_i = \{x \in X : \forall i \in I(x), \forall j \notin I(x) \sigma_i^{-1}(V_i(x_i)) > \sigma_j^{-1}(V_j(x_j)),$$

$$\forall i, j \in I(x) \sigma_i^{-1}(V_i(x_i)) = \sigma_j^{-1}(V_j(x_j))\}.$$

Due to continuity of ϕ we have, that there exists $t^* > 0$, such that for all $t \in [0, t^*)$ it follows $\phi(t, x, u) \in (\cap_{i \in I(x)} \partial M_i) \cup (\cup_{i \in I(x)} M_i)$.

Then, by definition of the derivative we obtain

$$\dot{V}(x) = \overline{\lim}_{t \rightarrow +0} \frac{1}{t} (V(\phi(t, x, u)) - V(x)) \quad (45)$$

$$= \overline{\lim}_{t \rightarrow +0} \frac{1}{t} \left(\max_{i \in I(x)} \{\sigma_i^{-1}(V_i(\phi_i(t, x, u)))\} - \max_{i \in I(x)} \{\sigma_i^{-1}(V_i(x_i))\} \right) \quad (46)$$

From the definition of $I(x)$ it follows that

$$\sigma_i^{-1}(V_i(x_i)) = \sigma_j^{-1}(V_j(x_j)) \quad \forall i, j \in I(x),$$

and therefore the index i , on which the maximum in $\max_{i \in I(x)} \{\sigma_i^{-1}(V_i(x_i))\}$ is reached, may be always set equal to the index on which the maximum $\max_{i \in I(x)} \{\sigma_i^{-1}(V_i(\phi_i(t, x, u)))\}$ is reached.

We continue estimates (45)

$$\dot{V}(x) = \overline{\lim}_{t \rightarrow +0} \max_{i \in I(x)} \left\{ \frac{1}{t} (\sigma_i^{-1}(V_i(\phi_i(t, x, u))) - \sigma_i^{-1}(V_i(x_i))) \right\}$$

Using Lemma 2 we obtain

$$\dot{V}(x) = \max_{i \in I(x)} \left\{ \overline{\lim}_{t \rightarrow +0} \frac{1}{t} (\sigma_i^{-1}(V_i(\phi_i(t, x, u))) - \sigma_i^{-1}(V_i(x_i))) \right\}.$$

Overall, we have that for all $x \in X$ holds

$$\frac{d}{dt} V(x) = \max_i \{ (\sigma_i^{-1})' (V_i(x_i)) \frac{d}{dt} V_i(x_i(t)) \} \leq -\alpha(V(x)),$$

and the theorem is proved for all $x \in X$. ■

Remark 6 In recent paper [20] it was proved a general vector small-gain theorem, that states roughly speaking that if the abstract control system possesses a vector ISS Lyapunov function, then it is ISS. The authors have also shown how from this theorem the small-gain theorems for interconnected systems of ODEs and retarded equations can be derived. It is possible, that the small-gain theorem, similar to the proved in this section, can be derived from the general theorem from [20]. However, it seems, that the constructions in [20] can be provided only for maximum formulation of ISS-Lyapunov functions (as in (36)). If the subsystems possess the ISS-Lyapunov functions in terms of summations, i.e. instead of (36) one has

$$V_i(x_i) \geq \sum_{j=1}^n \chi_{ij}(V_j(x_j)) + \chi_i(\|u\|_U) \Rightarrow \dot{V}_i(x_i) \leq -\alpha_i(V_i(x_i)), \quad (47)$$

then it is not clear, how the proofs from [20] can be adapted for this case. In contrast to it, the counterpart of the Theorem 3 in the summation case can be proved with the method, similar to the used in the proof of the Theorem 3, see [10]. However, the small-gain condition will have slightly another form.

6.1 Interconnections of linear systems

The construction of ISS-Lyapunov function for the interconnections of finite-dimensional input-to-state stable linear systems can be generalized to the case of interconnections of linear systems over Banach spaces.

Let X_i , $i = 1, \dots, n$ be Banach spaces. Consider n systems of the form

$$\dot{x}_i = A_i x_i(t), \quad i = 1, \dots, n, \quad (48)$$

where $x_i(t) \in X_i$, $A_i : X_i \rightarrow X_i$ is a generator of an analytic semigroup over X_i defined on $D(A_i) \subset X_i$.

Assume that all systems (48) are 0-UGAS and consider the following interconnection

$$\dot{x}_i = A_i x_i(t) + \sum_{j=1}^n B_{ij} x_j(t) + C_i u(t), \quad i = 1, \dots, n, \quad (49)$$

where $B_{ij} \in L(X_j, X_i)$, $i, j \in \{1, \dots, n\}$ are bounded operators, $u \in U_c = PC(\mathbb{R}_+, U)$ for some Banach space of input values U . We assume, that $B_{ii} = 0$, $i = 1, \dots, n$. Otherwise we can always substitute $\tilde{A}_i = A_i + B_{ii}$.

Let us denote $X = X_1 \times \dots \times X_n$ and introduce the matrix operators $A = \text{diag}(A_1, \dots, A_n) : X \rightarrow X$, $B = (B_{ij})_{i,j=1,\dots,n} : X \rightarrow X$ and $C = (C_1, \dots, C_n) : U \rightarrow X$. Then the system (49) can be rewritten in the following form

$$\dot{x}(t) = (A + B)x(t) + Cu(t). \quad (50)$$

Now we apply Lyapunov technique developed in this section to the system (49), for which we assume that A_i generates an analytic semigroup.

From Theorem 3 and Lemma 1 we have, that i -th subsystem of (49) is ISS iff the analytic semigroup generated by A_i is exponentially stable. This is equivalent (see [3], Theorem 5.1.3) to the existence of a corresponding Lyapunov function

$$V_i(x_i) = \langle P_i x_i, x_i \rangle, \quad (51)$$

where $P_i \in L(X_i)$ is a positive operator (linear self-adjoint bounded operator with $\langle P_i x_i, x_i \rangle \geq 0$ for all $x_i \in X_i$), for which it holds that

$$\langle A_i x_i, P_i x_i \rangle + \langle P_i x_i, A_i x_i \rangle \leq -\|x_i\|_{X_i}^2, \quad \forall x_i \in D(A_i). \quad (52)$$

Note that in [3], Theorem 5.1.3 this result was stated with $=$ instead of \leq , which is the equivalent formulation.

Differentiating V_i w.r.t. the i -th subsystem of (49), we obtain for all $x_i \in D(A_i)$

$$\begin{aligned} \dot{V}_i(x_i) &= \langle P_i \dot{x}_i, x_i \rangle + \langle P_i x_i, \dot{x}_i \rangle \leq \\ &(\langle P_i A_i x_i, x_i \rangle + \langle P_i x_i, A_i x_i \rangle) + 2\|x_i\|_{X_i} \|P_i\| \left(\sum_{j \neq i} \|B_{ij}\| \|x_j\|_{X_j} + \|C_i\| \|u\|_U \right). \end{aligned}$$

Operator P_i is self-adjoint, hence it holds $\langle P_i A_i x_i, x_i \rangle = \langle A_i x_i, P_i x_i \rangle$ and by equality (52) we obtain

$$\dot{V}_i(x_i) \leq -\|x_i\|_{X_i}^2 + 2\|x_i\|_{X_i}\|P_i\| \left(\sum_{i \neq j} \|B_{ij}\| \|x_j\|_{X_j} + \|C_i\| \|u\|_U \right).$$

Now take $\varepsilon \in (0, 1)$ and let

$$\|x_i\|_{X_i} \geq \frac{2\|P_i\|}{1-\varepsilon} \left(\sum_{i \neq j} \|B_{ij}\| \|x_j\|_{X_j} + \|C_i\| \|u\|_U \right). \quad (53)$$

Then we obtain for all $x_i \in D(A_i)$

$$\dot{V}_i(x_i) \leq -\varepsilon \|x_i\|_{X_i}^2.$$

To verify this inequality for all $x_i \in X_i$ we use the same argument, as in the end of the proof of the Theorem 2 (here we use analyticity of a semigroup).

We are going to rewrite the condition (53) in the form (36). To this end we mention that it holds

$$a_i^2 \|x_i\|_{X_i}^2 \leq V_i(x_i) \leq \|P_i\| \|x_i\|_{X_i}^2 \quad (54)$$

for some $a_i \in \mathbb{R}$.

The estimate from above is evident. To obtain the estimate from below, note that P_i is self-adjoint operator, thus if such a_i exists, then it can be chosen as $a_i^2 = \inf\{\text{Spec}(A_i)\}$ (see [21], p. 278). Operator P_i is positive, hence $\text{Spec}(P_i) \subset (0, +\infty)$. The spectrum of P_i is closed and we conclude that $a_i^2 > 0$.

To be able to apply the small-gain theorem, we replace inequality (53) with the following one

$$V_i(x_i) \geq \|P_i\|^3 \left(\frac{2}{1-\varepsilon} \right)^2 \left(\sum_{i \neq j} \frac{\|B_{ij}\|}{a_j} \sqrt{V_j(x_j)} + \|C_i\| \|u\|_U \right)^2. \quad (55)$$

It is easy to see that (55) together with (54) imply (53).

Thus, gains can be defined by:

$$\gamma_{ij}(s) = \left(\frac{2\|P_i\|^{3/2}}{1-\varepsilon} \frac{\|B_{ij}\|}{a_j} \right) \sqrt{s}, \quad (56)$$

for all $i \neq j$, $i = 1, \dots, n$. If the small-gain condition for this choice of gains holds, we can conclude the ISS of the system (50).

As a particular example consider the following system of interconnected linear reaction-diffusion equations

$$\begin{cases} \frac{\partial s_1}{\partial t} = c_1 \frac{\partial^2 s_1}{\partial x^2} + a_{12} s_2, & x \in (0, d), t > 0, \\ s_1(0, t) = s_1(d, t) = 0; \\ \frac{\partial s_2}{\partial t} = c_2 \frac{\partial^2 s_2}{\partial x^2} + a_{21} s_1, & x \in (0, d), t > 0, \\ s_2(0, t) = s_2(d, t) = 0. \end{cases} \quad (57)$$

We choose the state space as $X_1 = X_2 = L_2(0, d)$. The operators $A_i = c_i \frac{d^2}{dx^2}$ with $D(A_i) = H_0^1(0, d) \cap H^2(0, d)$, $i = 1, 2$ are generators of the analytic semigroups for the corresponding subsystems.

Both subsystems are ISS, moreover, $\text{Spec}(A_i) = \{-c_i \left(\frac{\pi n}{d}\right)^2 \mid n = 1, 2, \dots\}$, $i = 1, 2$.

Take $P_i = \frac{1}{2c_i} \left(\frac{d}{\pi}\right)^2 I$, where I is the identity operator on X_i . We have

$$\begin{aligned} \langle A_i s, P_i s \rangle + \langle P_i s, A_i s \rangle &= \frac{1}{c_i} \left(\frac{d}{\pi}\right)^2 \langle A_i s, s \rangle = \left(\frac{d}{\pi}\right)^2 \int_0^d \frac{\partial^2 s}{\partial x^2} s dx \\ &= - \left(\frac{d}{\pi}\right)^2 \int_0^d \left(\frac{\partial s}{\partial x}\right)^2 dx \leq -\|s\|_{L_2(0, d)}^2. \end{aligned}$$

In the last estimate we have used Friedrichs' inequality (see p. 67 in [24]). The Lyapunov functions for subsystems are defined by

$$V_i(s_i) = \langle P_i s_i, s_i \rangle = \frac{1}{2c_i} \left(\frac{d}{\pi}\right)^2 \|s_i\|_{L_2(0, d)}^2, \text{ for } s_i \in X_i.$$

We have the following estimates for derivatives

$$\begin{aligned} \dot{V}_1(s_1) &\leq -\|s_1\|_{L_2(0, d)}^2 + \frac{1}{c_1} \left(\frac{d}{\pi}\right)^2 |a_{12}| \|s_1\|_{L_2(0, d)} \|s_2\|_{L_2(0, d)}, \\ \dot{V}_2(s_2) &\leq -\|s_2\|_{L_2(0, d)}^2 + \frac{1}{c_2} \left(\frac{d}{\pi}\right)^2 |a_{21}| \|s_1\|_{L_2(0, d)} \|s_2\|_{L_2(0, d)}. \end{aligned}$$

We choose the gains in the following way

$$\gamma_{12}(r) = \frac{c_2}{c_1^3} \left(\frac{d}{\pi}\right)^4 \left| \frac{a_{12}}{1-\varepsilon} \right|^2 \cdot r, \quad \gamma_{21}(r) = \frac{c_1}{c_2^3} \left(\frac{d}{\pi}\right)^4 \left| \frac{a_{21}}{1-\varepsilon} \right|^2 \cdot r.$$

We have

$$\begin{aligned} V_1(s_1) \geq \gamma_{12} \circ V_2(s_2) &\Leftrightarrow \sqrt{\frac{c_1}{c_2} \gamma_{12}(1)} \|s_2\|_{L_2(0, d)} \leq \|s_1\|_{L_2(0, d)} \\ &\Leftrightarrow \frac{1}{c_1} \left(\frac{d}{\pi}\right)^2 |a_{12}| \|s_2\|_{L_2(0, d)} \leq (1-\varepsilon) \|s_1\|_{L_2(0, d)}. \end{aligned}$$

Analogously,

$$V_2(s_2) \geq \gamma_{21} \circ V_1(s_1) \Leftrightarrow \frac{1}{c_2} \left(\frac{d}{\pi}\right)^2 |a_{21}| \|s_1\|_{L_2(0, d)} \leq (1-\varepsilon) \|s_2\|_{L_2(0, d)}.$$

We have the following implications:

$$\begin{aligned} V_1(s_1) \geq \gamma_{12} \circ V_2(s_2) &\Rightarrow \dot{V}_1(s_1) \leq -\varepsilon \|s_1\|_{L_2(0, d)}^2, \\ V_2(s_2) \geq \gamma_{21} \circ V_1(s_1) &\Rightarrow \dot{V}_2(s_2) \leq -\varepsilon \|s_2\|_{L_2(0, d)}^2. \end{aligned}$$

The small-gain condition for the case of two interconnected systems can be equivalently written as $\gamma_{12} \circ \gamma_{21} < \text{Id}$ (see [8], p. 108).

$$\gamma_{12} \circ \gamma_{21} < \text{Id} \quad \Leftrightarrow \quad \frac{1}{c_1^2 c_2^2} \left(\frac{d}{\pi} \right)^8 \frac{|a_{12} a_{21}|^2}{(1 - \varepsilon)^4} < 1,$$

for arbitrary $\varepsilon > 0$. Thus, if

$$|a_{12} a_{21}| < c_1 c_2 \left(\frac{\pi}{d} \right)^4 \quad (58)$$

is satisfied, then the whole system (57) is 0-UGAS x .

6.2 Nonlinear example

Let us show the applicability of small-gain theorem to the nonlinear systems.

$$\begin{cases} \frac{\partial s_1}{\partial t} = c_1 \frac{\partial^2 s_1}{\partial x^2} + s_2^2, & x \in (0, d), \quad t > 0, \\ s_1(0, t) = s_1(d, t) = 0; \\ \frac{\partial s_2}{\partial t} = c_2 \frac{\partial^2 s_2}{\partial x^2} - b s_2 + \sqrt{|s_1|}, & x \in (0, d), \quad t > 0, \\ s_2(0, t) = s_2(d, t) = 0. \end{cases} \quad (59)$$

We assume, that c_1, c_2, b are positive constants and c_2 is close to zero.

Thus, we choose the state space and space of input values for the first subsystem as $X_1 = L_2(0, d)$, $U_1 = L_4(0, d)$ and for the second subsystem as $X_2 = L_4(0, d)$, $U_2 = L_2(0, d)$. The state of the whole system (59) is denoted by $X = X_1 \times X_2$.

Define operators $B_i = c_i \frac{d^2}{dx^2}$. These operators (together with Dirichlet boundary conditions) generate an analytic semigroup on $L_2(0, d)$ and $L_4(0, d)$ respectively (see, e.g. [26, Chapter 7]).

For both subsystems take the set of input functions as $U_{c,i} := C([0, \infty), U_i)$. We consider the mild solutions of the subsystems, that is the solutions s_i , given by the formula (6).

Note that $s_2 \in C([0, \infty), L_4(0, d)) \Leftrightarrow s_2^2 \in C([0, \infty), L_2(0, d))$ and $s_1 \in C([0, \infty), L_2(0, d)) \Leftrightarrow \sqrt{s_1} \in C([0, \infty), L_4(0, d))$.

Under made assumptions the solution of the first subsystem (when s_2 is treated as input) belongs to $C([0, \infty), H_0^1(0, d) \cap H^2(0, d)) \subset C([0, \infty), L_2(0, d))$ and the solution of the second one belongs to

$C([0, \infty), W_0^{4,1}(0, d) \cap W^{4,2}(0, d)) \subset C([0, \infty), L_4(0, d))$. This implies, that the solution of the whole system is from the space $C([0, T], X)$ for all T such that the solution of the whole system exists on $[0, T]$. The existence and uniqueness of the solution for all times will be proved for the values of parameters which establish ISS of the whole system, since this excludes the possibility of the blow-up phenomena.

Both subsystems of (59) are ISS. We choose V_i , $i = 1, 2$ defined by

$$V_1(s_1) = \int_0^d s_1^2(x) dx = \|s_1\|_{L_2(0,d)}^2, \quad V_2(s_2) = \int_0^d s_2^4(x) dx = \|s_2\|_{L_4(0,d)}^4$$

as ISS-Lyapunov functions for i -th subsystem.

Consider the Lie derivative of V_1 :

$$\begin{aligned} \frac{d}{dt}V_1(s_1) &= 2 \int_0^d s_1(x, t) \left(c_1 \frac{\partial^2 s_1}{\partial x^2}(x, t) + s_2^2(x, t) \right) dx \\ &\leq -2c_1 \left\| \frac{ds_1}{dx} \right\|_{L_2(0,d)}^2 + 2\|s_1\|_{L_2(0,d)}^2 \|s_2\|_{L_4(0,d)}^2 \end{aligned}$$

In the last estimation we have used Cauchy-Schwarz inequality. By Friedrichs' inequality, we obtain the estimation

$$\begin{aligned} \frac{d}{dt}V_1(s_1) &\leq -2c_1 \left(\frac{\pi}{d} \right)^2 \|s_1\|_{L_2(0,d)}^2 + 2\|s_1\|_{L_2(0,d)}^2 \|s_2\|_{L_4(0,d)}^2 \\ &= -2c_1 \left(\frac{\pi}{d} \right)^2 V_1(s_1) + 2\sqrt{V_1(s_1)}\sqrt{V_2(s_2)} \end{aligned}$$

Take

$$\chi_{12}(r) = \frac{1}{c_1^2 \left(\frac{\pi}{d} \right)^4 (1 - \varepsilon_1)^2} r, \quad \forall r > 0,$$

where $\varepsilon_1 \in (0, 1)$ - arbitrary constant. We obtain

$$V_1(s_1) \geq \chi_{12}(V_2(s_2)) \quad \Rightarrow \quad \frac{d}{dt}V_1(s_1) \leq -2\varepsilon_1 c_1 \left(\frac{\pi}{d} \right)^2 V_1(s_1).$$

Consider the Lie derivative of V_2 :

$$\begin{aligned} \frac{d}{dt}V_2(s_2) &= 4 \int_0^d s_2^3(x, t) \left(c_2 \frac{\partial^2 s_2}{\partial x^2}(x, t) - bs_2(x, t) + \sqrt{|s_1(x, t)|} \right) dx \\ &\leq -12c_2 \int_0^d s_2^2 \left(\frac{\partial s_2}{\partial x} \right)^2 dx - 4bV_2(s_2) + 4 \int_0^d s_2^3(x, t) \sqrt{|s_1(x, t)|} dx \end{aligned}$$

Applying for the last term Hölder inequality we obtain

$$\frac{d}{dt}V_2(s_2) \leq -4bV_2(s_2) + 4(V_2(s_2))^{3/4}(V_1(s_1))^{1/4}$$

Let

$$\chi_{21}(r) = \frac{1}{a^4(1 - \varepsilon_2)^4} r, \quad \forall r > 0,$$

where $\varepsilon_2 \in (0, 1)$ - arbitrary constant. It holds the implication

$$V_2(s_2) \geq \chi_{21}(V_1(s_1)) \quad \Rightarrow \quad \frac{d}{dt}V_2(s_2) \leq -4b\varepsilon_2 V_2(s_2).$$

The small-gain condition leads us to the following condition

$$\chi_{12} \circ \chi_{21} < \text{Id} \quad \Leftrightarrow \quad c_1^2 \left(\frac{\pi}{d} \right)^4 (1 - \varepsilon_1)^2 a^4 (1 - \varepsilon_2)^4 > 1 \quad \Leftrightarrow \quad c_1 \left(\frac{\pi}{d} \right)^2 a^2 > 1.$$

This condition guarantees that the system (59) is 0-UGAS x .

7 Conclusion

We have analyzed local and global ISS of infinite-dimensional control systems. We discussed the properties of linear ISS systems, and showed some differences between infinite- and finite-dimensional theory. For nonlinear systems with inputs Lyapunov methods and linearization principle have been developed. The small-gain theorem in terms of Lyapunov functions has been proved.

The results were illustrated on examples of linear and semilinear reaction-diffusion equations.

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References

1. Alekseev, V.M., Tikhomirov, V.M., Fomin, S.V.: Optimal control. Contemporary Soviet Mathematics. Consultants Bureau, New York (1987)
2. Cai, C., Teel, A.: Characterizations of input-to-state stability for hybrid systems. *Systems & Control Letters* **58**(1), 47–53 (2009)
3. Curtain, R.F., Zwart, H.: An introduction to infinite-dimensional linear systems theory, *Texts in Applied Mathematics*, vol. 21. Springer-Verlag, New York (1995)
4. Dashkovski, S.: Anisotropic function spaces and related semi-linear hypoelliptic equations. *Math. Nachr.* **248/249**, 40–61 (2003)
5. Dashkovskiy, S., Efimov, D., Sontag, E.: Input to state stability and allied system properties. *Automation and Remote Control* **72**, 1579–1614 (2011)
6. Dashkovskiy, S., Mironchenko, A.: On the uniform input-to-state stability of reaction-diffusion systems. In: Proceedings of the 49th IEEE Conference on Decision and Control, Atlanta, Georgia, USA, Dec. 15-17, 2010, pp. 6547–6552 (2010)
7. Dashkovskiy, S., Rüffer, B.S., Wirth, F.R.: An ISS Lyapunov function for networks of ISS systems. In: Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems (MTNS), Kyoto, Japan, July 24-28, 2006, pp. 77–82 (2006)
8. Dashkovskiy, S., Rüffer, B.S., Wirth, F.R.: An ISS small gain theorem for general networks. *Math. Control Signals Systems* **19**(2), 93–122 (2007)
9. Dashkovskiy, S.N., Rüffer, B.S.: Local ISS of large-scale interconnections and estimates for stability regions. *Systems & Control Letters* **59**(3-4), 241 – 247 (2010)
10. Dashkovskiy, S.N., Rüffer, B.S., Wirth, F.R.: Small Gain Theorems for Large Scale Systems and Construction of ISS Lyapunov Functions. *SIAM Journal on Control and Optimization* **48**(6), 4089–4118 (2010)
11. Engel, K.J., Nagel, R.: One-parameter semigroups for linear evolution equations, *Graduate Texts in Mathematics*, vol. 194. Springer-Verlag, New York (2000). With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt
12. Evans, L.C.: Partial Differential Equations, *Graduate Studies in Mathematics*, vol. 19. American Mathematical Society (1998)
13. Grüne, L.: Asymptotic behavior of dynamical and control systems under perturbation and discretization, *Lecture Notes in Mathematics*, vol. 1783. Springer-Verlag, Berlin (2002)
14. Hahn, W.: Stability of motion. Translated from the German manuscript by Arne P. Baartz. Die Grundlehren der mathematischen Wissenschaften, Band 138. Springer-Verlag New York, Inc., New York (1967)
15. Henry, D.: Geometric theory of semilinear parabolic equations, *Lecture Notes in Mathematics*, vol. 840. Springer-Verlag, Berlin (1981)

16. Hespanha, J.P., Liberzon, D., Teel, A.R.: Lyapunov conditions for input-to-state stability of impulsive systems. *Automatica J. IFAC* **44**(11), 2735–2744 (2008)
17. Hille, E., Phillips, R.S.: Functional analysis and semi-groups. American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I. (1996)
18. Jayawardhana, B., Logemann, H., Ryan, E.P.: Infinite-dimensional feedback systems: the circle criterion and input-to-state stability. *Commun. Inf. Syst.* **8**(4), 413–414 (2008)
19. Jiang, Z.P., Mareels, I.M.Y., Wang, Y.: A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica J. IFAC* **32**(8), 1211–1215 (1996)
20. Karafyllis, I., Jiang, Z.P.: A vector small-gain theorem for general non-linear control systems. *IMA Journal of Mathematical Control and Information* (2011). DOI doi: 10.1093/imamci/dnr001
21. Kato, T.: Perturbation theory for linear operators. *Classics in Mathematics*. Springer-Verlag, Berlin (1995). Reprint of the 1980 edition
22. Lin, Y., Sontag, E.D., Wang, Y.: A smooth converse Lyapunov theorem for robust stability. *SIAM J. Control Optim.* **34**(1), 124–160 (1996)
23. Mazenc, F., Prieur, C.: Strict Lyapunov functions for semilinear parabolic partial differential equations. *Mathematical Control and Related Fields* **1**, 231–250 (2011)
24. Mitrinović, D.S., Pečarić, J.E., Fink, A.M.: Inequalities involving functions and their integrals and derivatives, *Mathematics and its Applications (East European Series)*, vol. 53. Kluwer Academic Publishers Group, Dordrecht (1991)
25. Murray, J.D.: Mathematical biology. II, *Interdisciplinary Applied Mathematics*, vol. 18, third edn. Springer-Verlag, New York (2003). Spatial models and biomedical applications
26. Pazy, A.: Semigroups of linear operators and applications to partial differential equations, *Applied Mathematical Sciences*, vol. 44. Springer-Verlag, New York (1983)
27. Pepe, P., Jiang, Z.P.: A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems. *Systems Control Lett.* **55**(12), 1006–1014 (2006)
28. Rüffer, B.: Monotone dynamical systems, graphs, and stability of large-scale interconnected systems. Ph.D. thesis, Fachbereich 3 (Mathematik & Informatik) der Universität Bremen (2007)
29. Rüffer, B.S.: Monotone inequalities, dynamical systems, and paths in the positive orthant of Euclidean n -space. *Positivity*. **14**(2), 257–283 (2010)
30. Sontag, E.D.: Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control* **34**(4), 435–443 (1989)
31. Sontag, E.D.: Mathematical control theory, *Texts in Applied Mathematics*, vol. 6, second edn. Springer-Verlag, New York (1998). Deterministic finite-dimensional systems
32. Sontag, E.D.: Input to state stability: basic concepts and results. In: *Nonlinear and optimal control theory, Lecture Notes in Math.*, vol. 1932, pp. 163–220. Springer, Berlin (2008)
33. Sontag, E.D., Wang, Y.: On characterizations of the input-to-state stability property. *Systems Control Lett.* **24**(5), 351–359 (1995)
34. Sontag, E.D., Wang, Y.: New characterizations of input-to-state stability. *IEEE Trans. Automat. Control* **41**(9), 1283–1294 (1996)
35. Turing, A.M.: The Chemical Basis of Morphogenesis. *Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences* **237**(641), 37–72 (1952)
36. Vu, L., Chatterjee, D., Liberzon, D.: Input-to-state stability of switched systems and switching adaptive control. *Automatica J. IFAC* **43**(4), 639–646 (2007)
37. Willems, J.C.: Dissipative dynamical systems. I. General theory. *Arch. Rational Mech. Anal.* **45**, 321–351 (1972)